An Extended Discussion of the Committee Problem

From the *Mathematics Teacher* article *Both Answers Make Sense!*

In the article *Both Answers Make Sense!*, I discussed the Committees problem, which states, "How many ways can you form a 10-person committee from a group of 10 men and 10 women if at least 6 women must be on the committee?" There are two common answers to this problem. A correct response involves a case breakdown for committees that have 6, 7, 8, 9, or 10 women on the committee, respectively. This response yields the expression

$$\binom{10}{6}\binom{10}{4}+\binom{10}{7}\binom{10}{3}+\binom{10}{8}\binom{10}{2}+\binom{10}{9}\binom{10}{10}+\binom{10}{10}\binom{10}{0}.$$

We can get this answer by first choosing how many women are in the committee and then selecting men to fill out the rest of the committee. For any i women there are $\binom{10}{i}$ ways to choose those women, then there are $\binom{10}{i}$ ways to choose the remaining $10-i$ men to fill out the committee. Thus, for any number i there are $\binom{10}{i}\binom{10}{n-i}$ possible committees. Since we want committees with at least 6 women, we can sum over cases where $i = 6, 7, 8, 9, 10$.

Another common response, which is incorrect, is to argue, as Brady did in the article, for an answer of

$$\binom{10}{6}\binom{14}{4}.$$

The reasoning is that once the 6 women are chosen, there is no further restriction on how the remaining people are chosen, because we have satisfied the constraint that there are at least 6 women on the committee. Thus, we can choose any 4 of remaining 14 total people (10 men and 4 remaining women).

Again, as I note in the article, the problem is that this second answer is too big. The correct answer is 60,626, but the incorrect answer is 210,210. This gives a difference of 149,584. Although I gave an example in the article about why some outcomes get overcounted by the second answer, in this discussion I want to discuss more specifically which outcomes are overcounted (and how often they are overcounted). I hope that this discussion will supplement the article and will help readers reason through the details of how the overcounting works in this (and other) problems.

Which committees get overcounted?

The committees that are overcounted are those that have more than 6 women. We can consider the cases presented in the case breakdown above, and determine the number of overcounted solutions for each case. The table below shows how many committees are overcounted in each case, accounting for the total overcount. However, we must make a case for why the overcounting occurs at each case. I'll talk through the case of an 8-women committee to clarify this.

If we consider the 8-women case, notice that to count this number exactly, we will choose exactly 8 women, then choose exactly 2 men, which gives us

$$\binom{10}{8}\binom{10}{2} = 2025.$$

However, if we consider the alternative answer $\binom{10}{6}\binom{14}{4}$, we need to think about how many 8-women committees there are. Consider the outcome

$$\{W1, W2, W3, W4, W5, W6, W7, W8, M1, M2\}.$$

We want to think about every possible way we could generate this particular outcome with the counting process of first picking 6 women, and then picking any 4 people from the remaining 14 people. If this was allowed, we could have first picked the underlined women 1-6 in the first stage, and then gotten W7, W8, M1, and M2 in the next stage. The underlining shows that those are the women we pick in the first stage.

$$\{\underline{W1, W2, W3, W4, W5, W6}, W7, W8, M1, M2\}.$$

But, we also could have picked women 3-8 in the first stage, as seen below

$$\{\underline{W3, W4, W5, W6, W7, W8}, W1, W2, M1, M2\},$$

or we could have picked women 1, 2, 3, 5, 6, 7 in the first stage, as seen here:

$$\{\underline{W1, W2, W3, W5, W6, W7}, W4, W8, M1, M2\}.$$

The point is that all of these outcomes are really the SAME outcome, because we are counting committees, which comprise sets of people. The question now is, for this given committee involving women 1-8 and men 1-2, how many ways are there to generate the committee using the process $\binom{10}{6}\binom{14}{4}$? Another way to ask teh question is, given this particular outcome, how many ways are there to decide who goes in the underlined portion? If we think of it in this

way, it's not hard to see that we are generating this particular outcome a total of $8 choose 6$ times: one for each way that we could have a set of 6 women underlined. Thus, for any set that we have that consists of 8 women and 2 men, we have counted that set $8 choose 6$ different times. Therefore, using this incorrect process, we have a total of

$$\binom{10}{8}\binom{10}{2}\binom{8}{6} = 56,700$$

outcomes that have 8 women and 2 men. The problem is, we only wanted to count them

$$\binom{10}{8}\binom{10}{2} = 2025$$

times, and so we have overcounted the 8-women committees by 54, 675. This is just the 8-women committees. Similar arguments can be made for 7, 9, and 10-women committees. The table below highlights this information and shows exactly how we get an overcount of 149,584.

Number of Outcomes	Desired Outcomes	Outcomes Counted by Errant Process	Overcounted Outcomes
7	$\binom{10}{7}\binom{10}{3} = 14,400$	$\binom{10}{7}\binom{10}{3}\binom{7}{6} = 100,800$	86,400
8	$\binom{10}{8}\binom{10}{2} = 2,025$	$\binom{10}{8}\binom{10}{2}\binom{8}{6} = 56,700$	54,675
9	$\binom{10}{9}\binom{10}{1} = 100$	$\binom{10}{9}\binom{10}{1}\binom{9}{6} = 8,400$	8,300
10	$\binom{10}{10}\binom{10}{0} = 1$	$\binom{10}{10}\binom{10}{1}\binom{10}{6} = 210$	209
Total	60,626	210,210	149,584

Thus, we see that we can account for the overcounting for each case by examining how we may have actually chosen a certain number of women to be counted in the first stage of our counting process. I would recommend working through a smaller case of this problem in order to convince yourself how the overcounting actually occurs. Having students investigate how (especially in a smaller problem) might be a challenging exercise, but it would be an excellent way to help them to think carefully about their outcomes and about how a counting process generates outcomes. As an example, in order to organize this discussion, I worked on a smaller Committee problem: Suppose you have 5 women and 5 men, and you want to make a 5-person committee with at least 3 women. The outcomes here might be easier to examine, and I was able to find analogous formulas with the original problem that helped me get a sense of what was happening. Again, these activities are potentially powerful ways to reinforce one's sense of outcomes.